

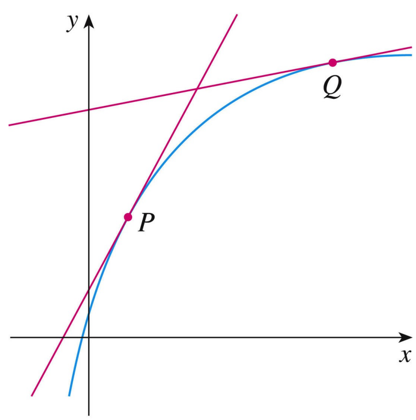
Lesson 18. Partial Derivatives

1 This lesson...

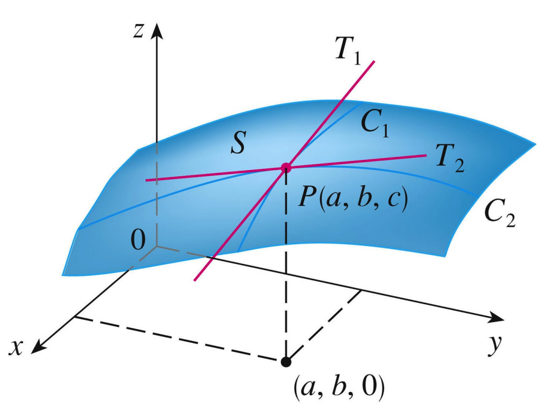
- Definition of partial derivative
- Computing partial derivatives
- Higher derivatives
- Practice!

2 Definition

- Derivatives of single-variable functions
    - Instantaneous rate of change
    - Slope of tangent line



- How can we get similar things for multivariable functions? Partial derivatives
- Idea: let $f(x, y)$ be a function of 2 variables
    - Fix the value of y to $b \Rightarrow g(x) = f(x, b)$ is a function in 1 variable x
    - Take the derivative of $g(x) = f(x, b)$ with respect to x
    - **This gives us the rate of change of $f(x, y)$ with respect to x when $y = b$**
    - Repeat, but with fixing the value of x and taking the derivative with respect to y



- The **partial derivative of $f(x, y)$ with respect to x** is

- The **partial derivative of $f(x, y)$ with respect to y** is

- In words, $\partial f / \partial x$ is

- In words, $\partial f / \partial y$ is

Example 1. Here is the wind-chill index function $W(T, v)$ from Lesson 16:

Wind speed (km/h)

Actual temperature (°C)

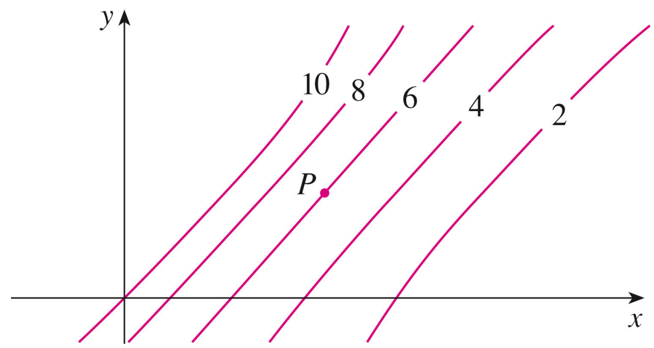
T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

a. Estimate $W_T(-15, 40)$.
b. Give a practical interpretation of this value.

Example 2. Here are the level curves for a function $f(x,y)$. Determine whether the following partial derivatives are positive or negative at the point P.

a. f_x

b. f_y



3 Computing partial derivatives

- Let $f(x,y)$ be a function of 2 variables
- To find f_x, treat y as a constant and differentiate $f(x,y)$ with respect to x
- To find f_y, treat x as a constant and differentiate $f(x,y)$ with respect to y

Example 3. Let $f(x,y) = 3x^3 + 2x^2y^3 - 5y^2$. Find $f_x(2,1)$ and $f_y(2,1)$.

Example 4. Let $f(x,y) = \frac{x}{y}$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Example 5. Let $f(x, y) = \sin\left(\frac{x}{1+y}\right)$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

4 Higher derivatives

- We can take partial derivatives of partial derivatives
- The **second partial derivatives of** $f(x, y)$ are
  - $f_{xx} =$
  - $f_{xy} =$
  - $f_{yx} =$
  - $f_{yy} =$
- **Clairaut's theorem.** Suppose f is defined on a disk D that contains the point (a, b).

  If f_{xy} and f_{yx} are continuous on D, then
- We can take third partial derivatives (e.g. f_{xxy}), fourth partial derivatives (e.g. f_{yxyy}), etc.

5 Examples

Do as many as you can!

Problem 1. Use the table of values of $f(x,y)$ to estimate the values of $f_x(3,2)$ and $f_y(3,2)$.

$x \backslash y$	1.8	2.0	2.2
2.5	12.5	10.2	9.3
3.0	18.1	17.5	15.9
3.5	20.0	22.4	26.1

Problem 2. Consider the level curves given in Example 2. Determine whether the following partial derivatives are positive or negative at the point P.

a. f_{xx}

b. f_{yy}

c. f_{xy}

Problem 3. Let $f(x,y) = \arctan(y/x)$. Find $f_x(2,3)$.

Problem 4. Let $f(x,y,z) = \dfrac{y}{x+y+z}$. Find $f_y(2,1,-1)$.

(Partial derivatives of functions of 3 or more variables are found the same way: regard all but one variable as constant, and take the derivative with respect to the remaining variable.)

Problem 5. Let $f(x, y, z) = \sqrt{\sin^2 x + \sin^2 y + \sin^2 z}$. Find $f_x(0, 0, \pi/4)$.

Problem 6. Find all the second partial derivatives of $f(x, y) = x^4 y - 2x^3 y^2$.

Problem 7. Let $f(x, y) = \cos(x^2 y)$. Verify that Clairaut's theorem holds: $f_{xy} = f_{yx}$.

Problem 8. Let $f(x, y) = \sin(2x + 5y)$. Find f_{yxy}.